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ON STEFAN'S PROBLEM OCCURRING IN THE THEORY OF POWDER BURNING

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A. I. SUSLOV
(Moscow)

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A one-dimensional single phase problem with a free boundary for the equation of heat conduction is studied. One of the conditions of the free boundary is nonlinear. Theorems of existence and uniqueness of solutions of the problem are proved. Sufficient conditions of stability of the solution under the perturbation of the initial parameters are derived for the case when the solution of the problem is of a stationary wave type.

We consider the following problem with a free boundary:

$$T_t = T_{xx} \quad \text{in the region} \tag{1}$$

$$D\ \{-\infty < x < s(t),\ 0 < t < A\}$$

$$T|_{t=0} = f(x), \quad T|_{x=s(t)} = 1, \quad T \xrightarrow[x\to-\infty]{} 0 \tag{2}$$

$$s'(t) = -u(p(t), \quad T_x(s(t), t)), \quad s(0) = 0 \tag{3}$$

The problem occurs in one-dimensional theory of powder burning at a constant surface temperature (see [1, 2]). The function $T(x, t)$ represents the temperature of the powder, u is the rate of combustion and $s(t)$ is the coordinate of the powder surface at the instance t. The functions $T(x, t)$ and $s(t)$ are to be determined.

The problem (1) — (3) differs from the classical Stefan's problem in the fact that the relation (3) connecting $s'(t)$ with $T_x(s(t), t)$ is nonlinear. The specified function $p(t)$ appearing in (3), describes the pressure in the gaseous phase $(x > s(t))$ containing the products of gasification of the powder and in the burning products.

Let us assume that $p(t) \in C[0, A]$, $p(t) > 0$; the function $u(p, \varphi)$, where φ is the temperature derivative at the boundary $s(t)$, is defined and continuous in the region $\{0 < p < \infty,\ 0 \leqslant \varphi < \infty\}$. Let also $0 \leqslant u(p, \varphi) \leqslant M_0$ for all p and φ.

Theorem 1. Let $f(x) \in C^2(-\infty, 0]$, f and $f' \to 0$ as $x \to -\infty$ and $f''(x)$ be a bounded function; $f(0) = 1$, $0 \leqslant f(x) \leqslant 1$. Then a solution $T(x, t)$, $s(t)$ of the problem (1) — (3) exists in the region D such that $T(x, t) \in C^{2,1}(D) \cap C(D \cup \partial D)$, $T_x \in C(D \cup \partial D)$, T and $T_x \to 0$ as $x \to -\infty$ uniformly in $t \in [0, A]$; $s(t) \in C^1[0, A]$.

Proof. We denote by G the region $\{-\infty < x < 0,\ 0 < t < A\}$, by H the closure of G, and consider in G the following boundary value problem:

$$T_t = T_{xx} - v(t)\, T_x \tag{4}$$

$$T|_{t=0} = f(x), \quad T|_{x=0} = 1, \quad T \xrightarrow[x\to-\infty]{} 0 \tag{5}$$

where $v(t) \in C[0, A]$, $0 \leqslant v(t) \leqslant M_0$. We formulate the auxiliary lemma.

Lemma 1. A unique solution $T(x, t)$ of the problem (4), (5) exists in G such that $T(x, t) \in C^{2,1}(G) \cap C^{1+\delta}(H)$, $0 \leqslant T(x, t) \leqslant 1$, T and $T_x \to 0$ as $x \to -\infty$ uniformly in $t \in [0, A]$.

Proof. Assume that
$$v(t) \in C^{\alpha/2}[0, A], \quad f''(x) \in C^{\alpha}(-\infty, 0] \tag{6}$$
with some constant $\alpha \in (0, 1)$, and that in addition
$$f''(0) - v(0) f'(0) = 0. \tag{7}$$
Then by Theorem 5.2, ch. 4 of [3] there exists in the region G a unique solution $T(x, t)$ of the problem (4), (5) belonging to the class $C^{2+\alpha}(H)$. By Theorem 10, ch. 3 of [4], Eq. (4) can be differentiated in the region G with respect to x, any number of times.

Using the maximum principle, we can show that $0 \leqslant T(x, t) \leqslant 1$, consequently $T_x(0, t) \geqslant 0$. We set $\Phi(x) = (x + l)^m / l^m$. By virtue of the properies of $f(x)$ the number $l > 0$ and the integer m can be chosen so that $f(x) \geqslant \Phi(x)$ for $-l \leqslant x \leqslant 0$. From the maximum principle it follows that $T(x, t) \geqslant \Phi(x)$ in the region $\{-l \leqslant x \leqslant 0, 0 \leqslant t \leqslant A\}$, provided that m is sufficiently large, therefore $T_x(0, t) \leqslant m / l$. Differentiating Eq. (4) with respect to x and applying the maximum principle, we obtain
$$|T_x| \leqslant \max\{\max|f'(x)|, m/l\} \tag{8}$$
When v and f satisfy the conditions (6) and (7), Eq. (4) holds in the closed region H, and $0 \leqslant T_{xx}(0,t) \leqslant mM_0 / l$. In the manner similar to the estimate (8), we obtain the inequality
$$|T_{xx}| \leqslant \max\{\sup|f''(x)|, mM_0/l\}. \tag{9}$$
Let us integrate the Green's identity for the functions $T(\xi, \tau) - 1$ and the fundamental solution
$$K(x, t; \xi, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\left(x - \xi - \int_{\tau}^{t} v(y)\,dy\right)^2 \Big/ 4(t-\tau)\right]$$
of Eq. (4) over the region $\{\varepsilon < \tau < t - \varepsilon, -\infty < \xi < 0\}$. Passing to the limit with $\varepsilon \to 0$, we obtain
$$T(x, t) - 1 = \int_0^t T_\xi(0, \tau) K(x, t; 0, \tau)\,d\tau + \int_{-\infty}^{0} [f(\xi) - 1] K(x, t; \xi, 0)\,d\xi \tag{10}$$
Differentiating (10) with respect to x and taking into account the fact that $K_x = -K_\xi$, we arrive at the equation
$$T_x(x, t) = \int_0^t T_\xi(0, \tau) K_x(x, t; 0, \tau)\,d\tau + \int_{-\infty}^{0} f'(\xi) K(x, t; \xi, 0)\,d\xi \tag{11}$$
From (8), (10) and (4) it follows that T, $T_x \to 0$ as $x \to -\infty$ uniformly in $t \in [0, A]$, provided that f and $f' \to 0$ as $x \to -\infty$. Further, by virtue of Theorem 4, ch. 7 of [4], the following inequality holds for any $\delta \in (0,1)$:
$$\|T\|_{C^{1+\delta}(H)} \leqslant B_1 \|f\|_{C^1(-\infty, 0]} \tag{12}$$
Here the constant B_1 depends on δ, M_0, A.

Thus we have proved the lemma under the additional restrictions (6), (7) imposed on $v(t)$ and $f(x)$.

In the general case we approximate the functions v and f with the functions $v_k(t)$ and $f_k(x)$ satisfying the conditions (6) and (7). Moreover, $0 \leqslant v_k(t) \leqslant M_0$, $v_k(t) \to v(t)$ as $k \to \infty$ uniformly on $[0, A]$, $f_k(x) \to f(x)$ as $k \to \infty$ over the norm $C^1(-\infty, 0]$.

Let us now consider the boundary value problem (4), (5) with $v_k(t)$ replacing $v(t)$ in Eq. (4) and $f_k(x)$ replacing $f(x)$ in the initial conditions (5). We have shown above that this problem has a solution $T_k \in C^{2+\alpha}(H)$ and the estimates (8), (9) and (12) hold for

T_k . From (9) it follows that the derivatives T_{kxx} are bounded by constants independent of k. Let us denote by G^ρ a set of points $(x, t) \in G$ separated from the boundary of G by a distance not less than ρ. Using the method of [5] one can estimate in G^ρ the derivatives T_{kxxx} using a constant depending on M_0, $\sup |T_{kxx}|$ and ρ only. The lemma 6 of [6] implies that in the region $G^{2\rho}$ the function T_{kxx} is Hölder continuous with the exponent equal to $^2/_3$ and Hölder constant independent of k. It follows therefore that a subsequence $k_n \to \infty$ can be found such that $T_{k_n} \to T$, $T_{k_n x} \to T_x$ uniformly in H and $T_{k_n xx} \to T_{xx}$ uniformly in the region $G^{2\rho} \cap \{x \geqslant -N\}$, where $N > 1$ is any number. It can be shown that $T(x, t)$ is a unique solution of (4), (5), T and $T_x \to 0$ as $x \to -\infty$ uniformly in $t \in [0, A]$ and in addition the estimate (12) holds for T. This completes the proof of Lemma 1.

Let $C^{1,0}(H)$ be a Banach space of functions continuous in H together with their derivatives in x, with the norm

$$\| T \|_{1,0} = \| T \|_{C(H)} + \| T_x \|_{C(H)}$$

We denote by V_K a set of functions $\theta \in C^{1,0}(H)$ such that $\theta(x, 0) = f(x)$, $\theta(0, t) = 1$, $0 \leqslant \theta(x, t) \leqslant 1$ and $\| \theta \|_{1,0} \leqslant K$, and consider in G a boundary value problem for the following equation with the boundary conditions (5)

$$T_t = T_{xx} - u(p(t), \theta_x(0, t))\, T_x, \quad \theta(x, t) \in V_K \tag{13}$$

According to Lemma 1, there exists a unique solution $T_\theta(x, t)$ of the problem (5), (13), $0 \leqslant T_\theta(x, t) \leqslant 1$ and the estimates (8) and (12) hold for T_θ. Therefore we can define on the set V_K a transformation F placing the function $\theta \in V_K$ in correspondence with the solution $T_\theta(x, t)$ of the problem (5), (13). From (8) it follows that for sufficiently large K we have the inclusion $FV_K \subset V_K$. Using (12) and the fact that $T_{\theta x} \to 0$ as $x \to -\infty$, we can show that the set FV_K is compact in $C^{1,0}(H)$. The operator F is continuous on V_K. Indeed, let $T_i = F\theta_i$, $u_i(t) = u(p(t), \theta_{ix}(0, t))$, $\theta_i \in V_K$, $i = 1,2$. We set $z = T_1 - T_2$. The function $z(x, t)$ satisfies the equation

$$z_t = z_{xx} - u_1(t)\, z_x - [u_1(t) - u_2(t)]\, T_{2x}$$

and the homogeneous boundary conditions. Therefore the following estimate holds:

$$\| z \|_{C^{1+\delta}(H)} \leqslant B_2 \| (u_1 - u_2)\, T_{2x} \|_{C(H)} \tag{14}$$

which is similar to the estimate (12). Here B_2 depends on δ, M_0, A. Let us assume that $\| \theta_1 - \theta_2 \|_{1,0} \leqslant \gamma$. Then

$$\max_{0 \leqslant t \leqslant A} | \theta_{1x}(0, t) - \theta_{2x}(0, t) | \leqslant \gamma$$

Since the function $u(p, \varphi)$ is continuous with respect to its arguments, then $| u_1(t) - u_2(t) | \leqslant \varepsilon(\gamma)$. Therefore the continuity of the operator F follows from the inequality (14).

The set V_K is closed and convex in $C^{1,0}(H)$. Consequently by the Schauder theorem the operator F has a fixed point $T_*(x, t)$ in V_K. The function $T_*(x, t)$ satisfies the equation

$$T_t = T_{xx} - u(p(t), T_x(0, t))\, T_x \tag{15}$$

and conditions (5). Let us set

$$s(t) = -\int_0^t u(p(\tau), T_{*x}(0, \tau))\, d\tau$$

and perform the change of variables $t = t$, $x' = x + s(t)$. The functions $T_*^\circ(x', t) = T_*(x' - s(t), t)$ and $s(t)$ represent a solution of the problem (1) — (3). This completes the proof of Theorem 1.

When $u(p, \varphi)$ are subjected to more severe restrictions, methods of [7] can be used to establish the theorem of existence and uniqueness of the solution of the problem (1)—(3), relaxing, at the same time, the restrictions imposed on $f(x)$:

Theorem 2. If $u(p, \varphi)$ is continuous in p and Lipschitz-continuous in φ, $0 \leqslant u \leqslant M_0$; $f(x) \in C^1(-\infty, 0]$, $0 \leqslant f(x) \leqslant 1$, $f(0) = 1$, $f(x) \to 0$ as $x \to -\infty$, then a unique solution $T(x, t)$, $s(t)$ of the problem (1) — (3) exists in the region D such that $T \in C^{2,1}(D) \cap C(D \cup \partial D)$, $s \in C^1[0, A]$.

If $p(t) \equiv 1$, $u(1, 1) = 1$ and $f(x) = e^x$, then the solution of (1) — (3) is of the stationary wave type $T = e^{x+t}$, $s(t) = -t$. The problem of stability of the stationary solution under the perturbations of the initial temperature distribution $f(x)$ is of interest. Below we formulate and prove a theorem which gives sufficient conditions of existence of a stationary solution under a constant pressure.

Theorem 3. If $p(t) \equiv 1$, $|f''(x)| \leqslant Me^{\alpha x}$ where $\alpha > 0$, the function $u(\varphi) \equiv u(1, \varphi)$ is Hölder-continuous and decreases monotonously in φ when $\varphi \geqslant 0$, then $u(T_x(s(t), t)) \to 1$ as $t \to \infty$, and the graph of $T(x, t)$ as a function of x assumes, as $t \to \infty$, the form of the graph of the function e^x as nearly as required.

Proof. It is sufficient to show that if $T(x, t)$ is a solution of the problem (5), (15), then $[T(x, t) - e^x] \to 0$ as $t \to \infty$ uniformly in $x \in (-\infty, 0]$ and $u(t) = u(T_x(0, t)) \to 1$ as $t \to \infty$.

Let $r = T(x, t) - e^x$, $q = r_x = T_x(x, t) - e^x$. The functions r and q satisfy, in the region G, the equations

$$r_t = r_{xx} - u(t)\, r_x - (u(t) - 1)\, e^x \tag{16}$$

$$q_t = q_{xx} - u(t)\, q_x - (u(t) - 1)\, e^x \tag{17}$$

In addition, we have $r(0, t) = 0$, $r \to 0$ as $x \to -\infty$.

We multiply (16) by r, (17) by q, sum the resulting expressions and integrate with respect to x from $-\infty$ to 0. Since

$$\int_{-\infty}^{0} [(u-1)\, e^x r + (u-1)\, e^x q]\, dx =$$

$$(u-1) \int_{-\infty}^{0} (e^x r + e^x r_x)\, dx = (u-1) \int_{-\infty}^{0} (e^x r)_x\, dx = 0$$

we have

$$\frac{\partial}{\partial t} \int_{-\infty}^{0} \frac{r^2 + r_x^2}{2}\, dx = - \int_{-\infty}^{0} (r_x^2 + r_{xx}^2)\, dx +$$

$$r_x(0, t)\, r_{xx}(0, t) - \frac{u}{2} r_x^2(0, t)$$

Since by virtue of (12) $u(\varphi)$ and $T_x(0, t)$ are Hölder-continuous with respect to their arguments, the function $u(t) = u(T_x(0, t))$ is Hölder-continuous in t. From the results of [8] it follows that in this case Eq. (15) holds in the closed regions $H \cap \{t \geqslant \rho\}$, $\rho > 0$. Consequently, $r_{xx}(0, t) = u(t)\, r_x(0, t) + u(t) - 1$ when $t > 0$ and

$$\frac{\partial}{\partial t}\int_{-\infty}^{0}\frac{r^2+r_x^2}{2}\,dx=-\int_{-\infty}^{0}(r_x^2+r_{xx}^2)\,dx+ \tag{18}$$
$$\frac{u}{2}r_x^2(0,t)+(u-1)r_x(0,t)$$

when $t>0$. Using the maximum principle we can obtain, from the conditions $|f''(x)|\leqslant Me^{\alpha x}$ the inequalities $|T,\ T_x,\ T_{xx}|\leqslant M_1e^{\alpha x}$. Therefore all integrals appearing in (18) are meaningful and the order of differentiation with respect to t and integration with respect to x can be reversed.

We shall show that the right-hand side of (18) is nonpositive. Since the function u (φ) decreases monotonously and $u(1)=1$ for all $t\geqslant 0$, inequality $(u(t)-1)\,r_x(0,t)\leqslant 0$ holds. Let us estimate the term $u(t)\,r_x^2(0,\ t)/2$ using other terms with the corresponding sign. If $u(t_1)>2$, then $0\leqslant T_x(0,t_1)<1$, consequently $-1\leqslant r_x(0,t_1)<0$. From this we have

$$\frac{u(t_1)}{2}r_x^2(0,t_1)+(u(t_1)-1)r_x(0,t_1)\leqslant \tag{19}$$
$$|r_x(0,t_1)|\left(\frac{u(t_1)}{2}-u(t_1)+1\right)=|r_x(0,t_1)|\left(1-\frac{u(t_1)}{2}\right)<0$$

Let $u(t_2)\leqslant 2$. Since

$$\frac{u(t)}{2}r_x^2(0,t)=\int_{-\infty}^{0}u(t)\,r_x(x,t)\,r_{xx}(x,t)\,dx$$

we have

$$\frac{u(t_2)}{2}r_x^2(0,t_2)-\int_{-\infty}^{0}[r_x^2(x,t_2)+r_{xx}^2(x,t_2)]\,dx= \tag{20}$$
$$-\int_{-\infty}^{0}(r_x^2-ur_xr_{xx}+r_{xx}^2)\big|_{t=t_2}dx\leqslant 0$$

From (19) and (20) the nonpositiveness of the right-hand side of (18) follows. Let us now set

$$g(t)=\int_{-\infty}^{0}\frac{r^2+r_x^2}{2}\,dx$$

Since $g(t)\geqslant 0$ and, as was shown above, $g'(t)\leqslant 0$ for all $t>0$, it follows that $\lim\limits_{t\to\infty}g(t)=g_0\geqslant 0$ exists.

Suppose that $g_0>0$. Since $r(0,\ t)=0$, the following inequality holds:

$$\int_{-\infty}^{0}r^2dx\leqslant\left(\frac{3}{2}\right)^{2/3}\left(\int_{-\infty}^{0}|r|\,dx\right)^{4/3}\left(\int_{-\infty}^{0}r_x^2dx\right)^{1/3} \tag{21}$$

The inequality (21) can be proved in the manner analogous to that of Theorem 2. 2, ch. 3, of [3]. Since $0\leqslant T\leqslant M_1e^{\alpha x}$, we have

$$\int_{-\infty}^{0}|r|\,dx<B_3$$

where B_3 is independent of t. Therefore from (21) it follows that

$$\int_{-\infty}^{0}r_x^2dx\geqslant g_1>0$$

for all $t>0$, provided that $g_0>0$.

Let us separate the half-line $\{t > 0\}$ into three sets: $S_1 = \{t: u(t) \geqslant 2\}$, $S_2 = \{t: {}^3/_2 < u(t) < 2\}$, $S_3 = \{t: u(t) \leqslant {}^3/_2\}$. The following inequality holds for $t \in S_1$:

$$g'(t) \leqslant -\int_{-\infty}^{0} r_x^2 dx \leqslant -g_1 < 0 \tag{22}$$

From a property of $u(\varphi)$ it follows that $r_x(0, t) < -g_2 < 0$, provided that ${}^3/_2 < u(t) < 2$. Taking into account (20), we obtain

$$g'(t) \leqslant (u-1) r_x(0, t) \leqslant -g_2 / 2 \tag{23}$$

for $t \in S_2$. When $t \in S_3$, we have

$$g'(t) \leqslant -\frac{7}{16} \int_{-\infty}^{0} r_x^2 dx \leqslant -\frac{7}{16} g_1 < 0 \tag{24}$$

The inequalities (22)—(24) were obtained under the assumption that $g_0 > 0$. It follows from these inequalities that $g'(t) < -g_3 < 0$ for all $t > 0$ and that $g \to -\infty$ as $t \to \infty$. Thus the assumption that $g_0 > 0$ leads to contradiction. Therefore $g_0 = 0$ and

$$\lim_{t \to \infty} \int_{-\infty}^{0} r^2 dx = 0, \quad \lim_{t \to \infty} \int_{-\infty}^{0} r_x^2 dx = 0$$

Since

$$r^2 = -\int_{x}^{0} 2r r_x dx \leqslant \int_{-\infty}^{0} (r^2 + r_x^2) dx$$

$r \to 0$ as $t \to \infty$ uniformly in x. The boundedness of the derivative r_{xx} and the fact that $r(0, t) = 0$, imply that $r_x(0, t) \to 0$ as $t \to \infty$. Therefore $T_x(0, t) \to 1$ and $u(t) \to 1$ as $t \to \infty$. This completes the proof of Theorem 3.

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